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Comments on 'Radial pulsations of a fluid sphere in a sound wave' by S. Temkin

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An error in a recent paper on bubble and drop oscillations by Temkin (*J. Fluid Mech.* vol. 380 (1999), pp. 1–38) is pointed out and corrected. In this way, his results are shown essentially to reduce to earlier ones in the literature. A concise derivation of these earlier results is presented for the case of a gas bubble.

In several earlier papers on the oscillations of gas bubbles in liquids (e.g. Prosperetti 1977, 1991) it was shown that the behaviour of the gas in the bubble gradually changes from isothermal to adiabatic as, with increasing frequency, the thermal penetration depth becomes smaller than the bubble radius. In a recent paper, Temkin (1999) questions these results arguing that the gas remains essentially isothermal. The purpose of the present note is to point to an error in Temkin's analysis, the correction of which confirms the validity of the earlier results for which independent derivations (Fanelli, Prosperetti & Reali 1981*a,b*) as well as experimental evidence (Crum 1983) are available. We use Temkin's notation throughout.

That Temkin's result must be in error is already apparent from his figure 2 which, near the bubble resonance frequency, shows the temperature of the gas–liquid interface higher than the mean gas temperature. Given that the only part of the system that can possibly undergo a significant temperature change is the gas, this result is physically difficult to understand. Temkin's figure 2 is essentially a consequence of his equation (3.29). Tracing back the origin of this equation, one is led to Temkin's expression (3.16*b*) for the constant D:

$$D = \frac{\beta_p \kappa_p T_s'}{j_0(q_i)}, \tag{1}$$

where T_s' denotes the temperature perturbation of the bubble surface, β_p is the gas coefficient of thermal expansion, and

$$k_i = \frac{\omega}{c_{sp}}\left[1 + O\left(\frac{k_i}{K_i}\right)^2\right], \quad K_i^2 = i\frac{\omega}{\kappa_p}\left[1 + O\left(\frac{k_i}{K_i}\right)^2\right], \tag{2}$$

with c_{sp} and κ_p respectively the speed of sound and thermal diffusivity in the gas. The omission of terms of order $|k_i/K_i|^2$ essentially amounts to neglecting the square of the ratio of the thermal penetration depth to the sound wavelength in the gas and is therefore well justified whenever a spherical model for the bubble is appropriate.‡

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‡ A spherical model will be justified provided the wavelength is large compared with the bubble radius a. If, on this basis, we estimate the maximum value of k_i as $k_i \sim 2\pi/a$ and of ω by $\omega \sim 2\pi c_{sp}/a$ we find that $|k_i/K_i|^2 \sim 2\pi\kappa_p/(ac_{sp})$. For a 100 µm air bubble in water this ratio is of the order of 3.7×10^{-3}.

The expression (1) for D is an approximation; the exact value found by following Temkin's analysis is

$$D = \frac{\beta_p \kappa_p T_s' + (\gamma_p - 1)(k_i^2/K_i^2)[p_s'/(\mathrm{i}\rho_{p0}\omega)]}{j_0(q_i)[1 + (\gamma_p - 1)(k_i^2/K_i^2)]}, \tag{3}$$

where p_s' is the pressure perturbation at the bubble surface. The second term in the denominator is of the same order as the terms omitted in (2) and is negligible with respect to 1; Temkin however also drops the second term in the numerator even though it is multiplied by a relatively large quantity. For example, if one assumes the validity of the perfect-gas laws, it is easy to show that the ratio of the second to the first term in the numerator of (3) is

$$\frac{\beta_p \kappa_p T_s'}{(\gamma_p - 1)(k_i^2/K_i^2)[p_s'/(\mathrm{i}\rho_{p0}\omega)]} = \frac{\gamma_p}{\gamma_p - 1} \frac{T_s'/T_0}{p_s'/p_0}. \tag{4}$$

This result shows that dropping the second term amounts to assuming that $|T_s'|/T_0$ is much larger than $|p_s'|/p_0$ which is actually the opposite of the actual situation due to the large heat capacity of the liquid relative to the gas.

The consequence of Temkin's use of the incorrect expression (1) is his equation (3.29)

$$T_s' = -\tfrac{1}{3} q_i^2 G(q_i) \bar{T}_p' \tag{5}$$

(where $\bar{T}_p'$ is the average temperature perturbation in the bubble) rather than the correct result

$$T_s' = \tfrac{1}{3} b_i^2 G(b_i) \frac{[(\gamma_p - 1)/\gamma_p](T_0/p_0)(1 - H)\bar{p}_p' - [1 + (\gamma_p - 1)(k_i/K_i)^2 H]\, \bar{T}_p'}{[1 + (\gamma_p - 1)(k_i/K_i)^2 H][H + (k_i/K_i)^2(\gamma_p - 1)] - (\gamma_p - 1)(k_i/K_i)^2(1 - H)^2}, \tag{6}$$

where

$$H = \frac{b_i^2}{q_i^2} \frac{G(b_i)}{G(q_i)}. \tag{7}$$

From this point on the analysis can be continued as in Temkin's paper. In particular, for the case of a 100 μm radius air bubble in water considered in Temkin's figure 7, the polytropic index behaves as shown in figure 1, where the dashed line is Temkin's result and the solid line the present corrected one. As expected, the polytropic index is close to 1 at low frequencies and increases with frequency. The normalized bubble surface temperature disturbance $T_s'/\bar{T}_p'$ can also be calculated following Temkin's procedure with the corrected expression for D and is shown in figure 2 (solid line); as expected both physically and on the basis of *a priori* quantitative estimates (Kamath, Prosperetti & Egolfopoulos 1993), T_s' is very small compared to $\bar{T}_p'$ in opposition to Temkin's result (dashed line) (even though, at low frequencies, $|T_s'/\bar{T}_p'|$ is as large as 10%, $\bar{T}_p'$, is exceedingly small and both quantities are essentially negligible).

In conclusion it may be of some interest to point out that, if one is willing to make the approximation $T_s' \ll T_0$ at the outset, the analysis becomes extremely simple; we present the calculation for a perfect gas.

After linearization of the conservation equations, the pressure and temperature disturbances in the bubble are given by (see e.g. equations (2.12) and (2.13*b*) in Temkin)

$$p_p' = \mathrm{i}\rho_{p0}\omega[\phi_1(k_i r) + \phi_2(K_i r)], \quad -\kappa_p \frac{T_p'}{T_0} = (\gamma_p - 1)\frac{k_i^2}{K_i^2}\phi_1(k_i r) - \phi_2(K_i r), \tag{8}$$

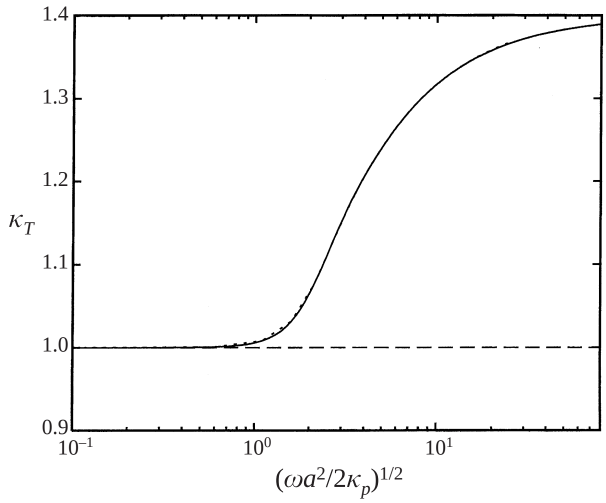

FIGURE 1. The gas polytropic exponent in a 100 μm radius air bubble in water undergoing forced volume pulsations at an angular frequency ω according to the present corrected theory (solid line) and to Temkin's original theory (dashed line); a is the bubble equilibrium radius and κ_p the gas thermal distinguishible. The barely distinguishable dotted line is the approximate result (16).

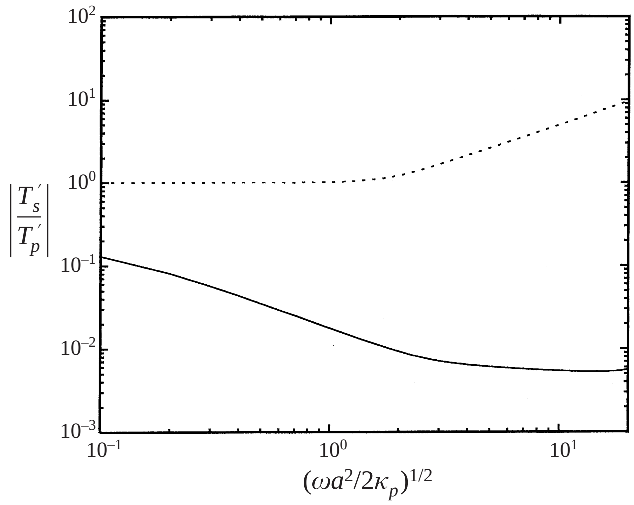

FIGURE 2. Ratio of the temperature perturbation T_s' at the bubble surface to the volume average of the temperature perturbation in the bubble $\bar{T}_p'$; ω is the angular frequency of the driving pressure perturbation, a is the bubble equilibrium radius, and κ_p is the gas thermal diffusivity. The solid line is the present result and the dashed line Temkin's.

where $\phi_{1,2}$ satisfy a Helmholtz equation with wavenumbers k_i and K_i respectively.

Temkin defines a polytropic index κ_T on the basis of the volume-averaged linearized pressure–temperature relation

$$\frac{\bar{T}_p'}{T_0} = \frac{\kappa_T - 1}{\kappa_T}\frac{\bar{p}_p'}{p_0}. \tag{9}$$

Since both $\bar{T}_p'$ and $\bar{p}_p'$ are complex with different phases, the actual definition of κ_T is

$$\kappa_T = \mathrm{Re}\left(\frac{1}{1-(\bar{T}_p'/T_0)/(\bar{p}_p'/p_0)}\right), \tag{10}$$

with the imaginary part related to energy dissipation.

The volume average of the expressions (8) for the pressure and temperature disturbances can be written in a convenient form by using the Helmholtz equations satisfied by $\phi_{1,2}$ together with the divergence theorem to find

$$\bar{\phi}_1 = -\frac{3}{ak_i}\phi_1'(k_i a), \tag{11}$$

where $\phi_1'(z) = \mathrm{d}\phi_1/\mathrm{d}z$; the corresponding expression for $\bar{\phi}_2$ is similar, with K_i in place of k_i. Upon using these representations for the averages, we have from (8)

$$\frac{\bar{T}_p'/T_0}{\bar{p}_p'/p_0} = \frac{1}{\gamma_p}\frac{(\gamma_p-1)\phi_1' - (K_i/k_i)\,\phi_2'}{\phi_1' + (k_i/K_i)\,\phi_2'}. \tag{12}$$

Let now

$$H = \frac{k_i}{K_i}\frac{\phi_1(k_i a)}{\phi_1'(k_i a)}\frac{\phi_2'(K_i a)}{\phi_2(K_i a)}, \tag{13}$$

which is readily seen to coincide with the previous definition (7), and note that, upon setting to zero the temperature disturbance at the bubble surface, the second of (8) gives

$$\frac{\phi_2(K_i a)}{\phi_1(k_i a)} = (\gamma_p - 1)\frac{k_i^2}{K_i^2}. \tag{14}$$

Then (12) may be identically rewritten as

$$\frac{\bar{T}_p'/T_0}{\bar{p}_p'/p_0} = \frac{\gamma_p - 1}{\gamma_p}\frac{1-H}{1+(\gamma_p-1)(k_i^2/K_i^2)\,H}, \tag{15}$$

in which the last term in the denominator should be dropped for consistency with (2). Upon substituting into (10) we thus find

$$\kappa_T = \mathrm{Re}\left(\frac{\gamma_p}{1+(\gamma_p-1)H}\right). \tag{16}$$

The potentials ϕ_1 and ϕ_2 are solutions of the spherically symmetric Helmholtz equation regular at the centre of the bubble with wavenumbers k_i and K_i respectively and, therefore,

$$H = \frac{k_i^2}{K_i^2}\frac{K_i a\,\cot(K_i a) - 1}{k_i a\,\cot(k_i a) - 1}. \tag{17}$$

However, since the spherical model only makes sense when the wavelength of the pressure disturbance is much greater than the bubble radius so that $k_i a \ll 1$, one may also use the approximation

$$H = \frac{3}{K_i^2 a^2}\left[1 - K_i a\,\cot(K_i a)\right], \tag{18}$$

from which

$$\kappa_T \simeq \mathrm{Re}\left(\frac{\gamma_p (K_i a)^2}{(K_i a)^2 + 3(\gamma_p - 1)[1-(K_i a)\cot(K_i a)]}\right), \tag{19}$$

which is equivalent to the form given in the earlier papers (e.g. Prosperetti 1977, 1991; Prosperetti, Crum & Commander 1988). From this form one readily deduces the low- and high-frequency limits $\kappa_T \simeq 1$ and $\kappa_T \to \gamma_p$, respectively. The result (16) is shown by the dotted line in figure 1, where it is barely distinguishable from the exact result obtained using Temkin's procedure with the correct value of D.

The definition (9) of the polytropic index used by Temkin is not the only possible one. The definition used in our earlier papers is based on the linearized pressure–volume relation

$$\kappa_V = -\frac{V_0}{p_0}\,\mathrm{Re}\left(\frac{\bar{p}'_p}{V'}\right), \tag{20}$$

in which V denotes the bubble volume. In order to calculate this quantity note that

$$-\mathrm{i}\omega V' = \frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi a^2 \left.\frac{\partial}{\partial r}(\phi_1 + \phi_2)\right|_{r=a}, \tag{21}$$

so that

$$\kappa_V = \frac{\mathrm{i}\omega a}{3p_0}\,\frac{\bar{p}'_p}{\partial(\phi_1+\phi_2)/\partial r}. \tag{22}$$

Proceeding as before one finds

$$\kappa_V = \gamma_p \frac{1 + (\gamma_p - 1)(k_i/K_i)^2 H}{1 + (\gamma_p - 1)H}, \tag{23}$$

which, upon neglecting the second term in the numerator, coincides with (16).

The error due to the use of the incorrect form (1) of the constant D also affects the expression for the thermal damping of the bubble but, since this matter was adequately treated in our earlier work, we do not pursue it here.

Temkin's theory is developed treating at the same time the case of bubbles in liquids specifically considered above, and also droplets in gases and droplets in an immiscible liquid. In principle, of course, the same error affects all these results although its consequences in the liquid–liquid system would be less severe due to the smaller thermal effects in that case.

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